Adama Science and Technology University

School of Electrical Engineering and Computing

Department of Electrical Power and Control Engineering

Fundamentals of Electrical Engineering (EPCE210)

Chapter Three Handout On

Circuit Theorems

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CHAPTER THREE

CIRCUIT THEOREMS

3.1 INTRODUCTION

The growth in areas of application of electric circuits has led to an evolution from simple to complex circuits. To handle the complexity, engineers over the years have developed some **theorems to simplify circuit analysis**. Such theorems include **Thevenin's and Norton's theorems**. Since these theorems are applicable to linear circuits, we first discuss the concept of circuit linearity. In addition to circuit theorems, *we discuss the concepts of superposition, maximum power transfer.*

3.2 LINEARITY PROPERTY

Linearity is the property of an element describing a linear relationship between cause and effect. The property is a combination of both the homogeneity property and the additivity property.

The homogeneity property requires that if the input (also called the excitation) is multiplied by a constant, then the output (also called the response) is multiplied by the same constant. For a resistor, for example, Ohm's law relates the input i to the output v,

$$v = iR \tag{3.1}$$

If the current is increased by a constant **k,** then the voltage increases correspondingly by k**,** that is,

$$kiR = kv \tag{3.2}$$

The additivity property requires that the response to a sum of inputs is the sum of the responses to each input applied separately. Using the relationship of a resistor, if

$$v_1 = i_1R \quad \text{and} \quad v_2 = i_2R \tag{3.3}$$

then applying $(i_1 + i_2)$ gives

$$v = (i_1 + i_2)\,R = i_1R + i_2R = v_1 + v_2 \tag{3.4}$$

A linear circuit is one whose output is linearly related (or directly proportional) to its input.

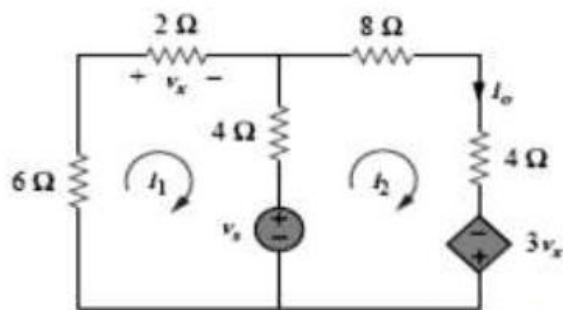Example 3.1: For the circuit in **Fig. 3.1**, find i_o when v_s = 12 V and v_s = 24 V.

Figure 3.1 For Example 3.1.

Solution: Applying KVL to the two loops, we obtain

$$12i_1 - 4i_2 + v_s = 0 \qquad\qquad \textbf{(3.1.1)}$$

$$-4i_1 + 16i_2 - 3v_x - v_s = 0 \qquad\qquad \textbf{(3.1.2)}$$

But $v_x = 2i_1$. Equation (3.1.2) becomes

$$-10i_1 + 16i_2 - v_s = 0 \qquad\qquad \textbf{(3.1.3)}$$

Adding Eqs. (3.1.1) and (3.1.3) yields

$$2i_1 + 12i_2 = 0 \Rightarrow i1 = -6i_2$$

Substituting this in Eq. (3.1.1), we get
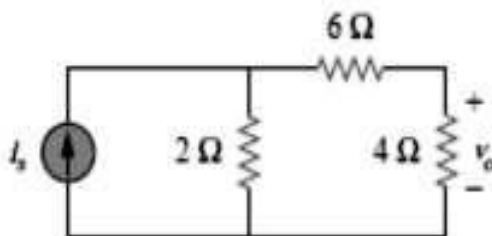
$$-76i_2 + v_s = 0 \Rightarrow i_2 = v_s/76$$

When v_s = 12V, $i_o = i_2 = 12/76$A

When v_s = 24 V, $i_o = i_2 = 24/76$A

showing that when the **source value is doubled**, *I_o doubles.*

Practice problems:

1-For the circuit in Figure below, find v_o when i_s = 15 and i_s = 30 A.

2

3.3 SUPERPOSITION

The idea of superposition rests on the linearity property.

The superposition principle states that the voltage across (or current through) an element in a linear circuit is the algebraic sum of the voltages across (or currents through) that element due to each independent source acting alone.

However, to apply the superposition principle, we must keep two things in mind:

1. We consider one independent source at a time while all other independent sources are turned off. This implies that we replace every voltage source by 0 V (or a short circuit), and every current source by 0 A (or an open circuit).

2. Dependent sources are left intact because they are controlled by circuit variables. With these in mind, we apply the superposition principle in three steps:

Steps to Apply Super position Principle:

1. Turn off all independent source except one source. Find the output (voltage or current) due to that acting source using nodal or mesh analysis.
2. Repeat step 1 for each of the other independent sources.
3. Find the total contribution by adding algebraically all the contributions due to the independent sources.

Analyzing a circuit using superposition has one major disadvantage: it may very likely involve more work. Keep in mind that superposition is based on linearity.
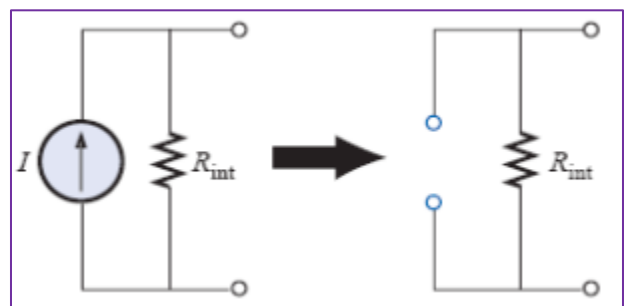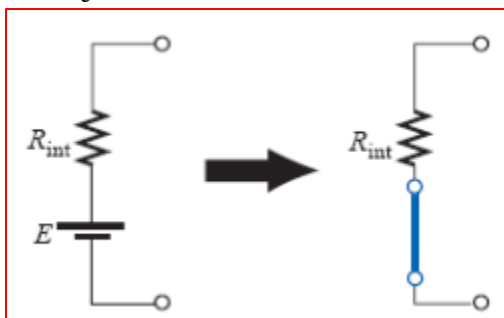
How removing the effects of practical sources

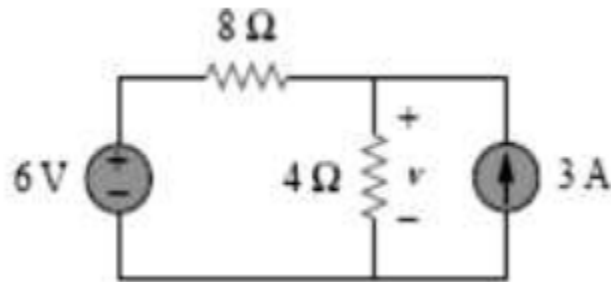Example 3.2: Use the superposition theorem to find v in the circuit in **Fig. 3.2.**

Figure 3.2 For Example 3.2.

Solution:

$$v = v_1 + v_2$$

where v_1 and v_2 are the contributions due to the 6-V voltage source and the 3-A current source, respectively. To obtain v_1, we set the current source to zero, as shown in Fig. 3.3(a).
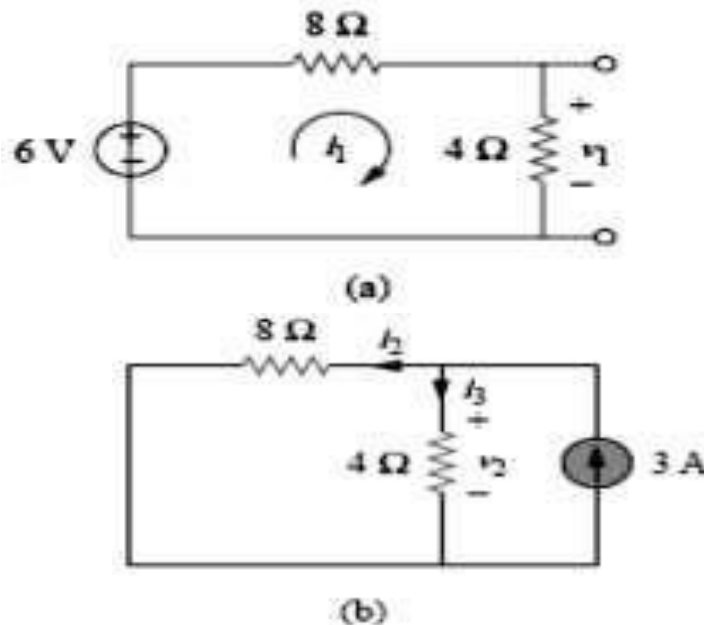

(a)

(b)

Figure 3.3 For Example 3.2: (a) calculating v_1, (b) calculating v_2.

Applying KVL to the loop in Fig. 3.3(a) gives

$$12i_1 - 6 = 0 \Rightarrow i_1 = 0.5 \text{ A}$$

Thus,

$$v_1 = 4i_1 = 2 \text{ V}$$

We may also use voltage division to get **v₁** by writing

$$v_1 = \frac{4}{4+8}(6) = 2\,V$$

To get v_2, we set the voltage source to zero, as in Fig. 3.3(b).

Using current division,

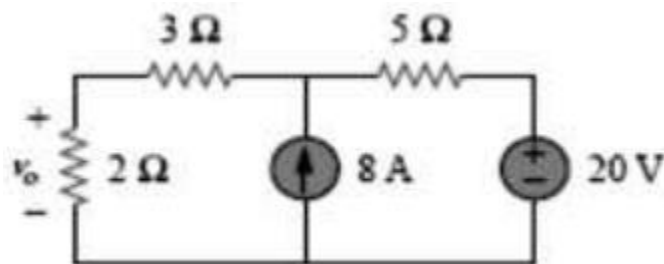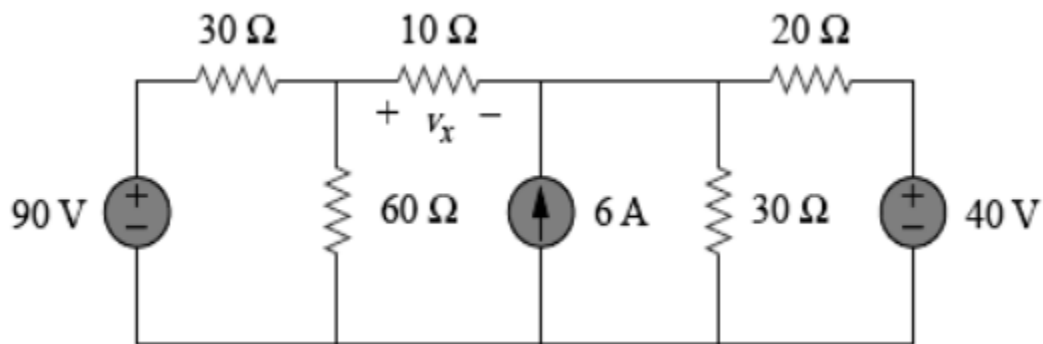$$i_3 = \frac{8}{4+8}(3) = 2\,A$$

Hence, $v_2 = 4i_3 = 8\,V$

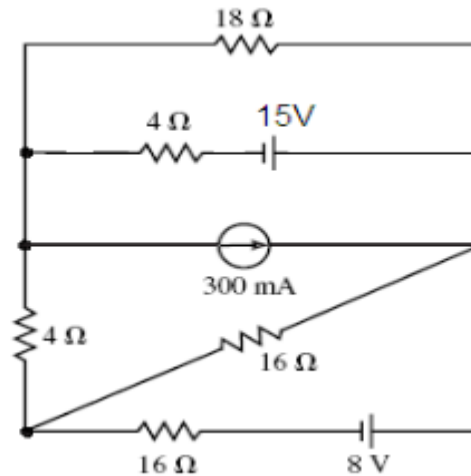And we find   **v = v₁ + v₂ = 2 + 8 = 10 V**

Practice problems:

1-Using the superposition theorem, find v_o in the circuit in Figure below.

2- Use superposition to obtain v_x in the circuit of Figure below.

3- Using superposition theorem determines current through and voltage across each

4.4 SOURCE TRANSFORMATION

We have noticed that series-parallel combination and wye-delta transformation help simplify circuits. Source transformation is another tool for simplifying circuits. We can substitute a voltage source in series with a resistor for a current source in parallel with a resistor, or vice versa, as shown in Fig. 3.4. Either substitution is known as a **source transformation**.
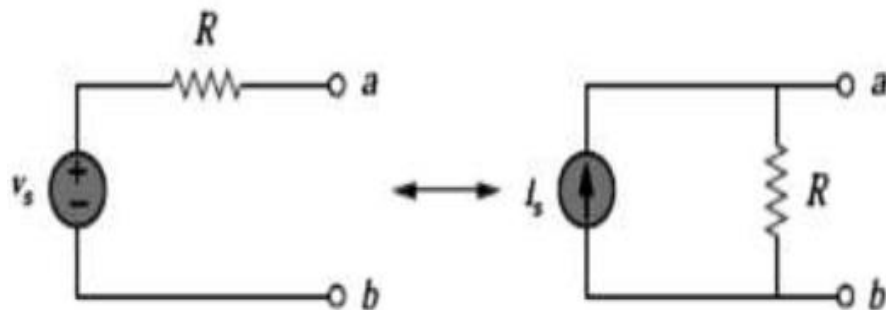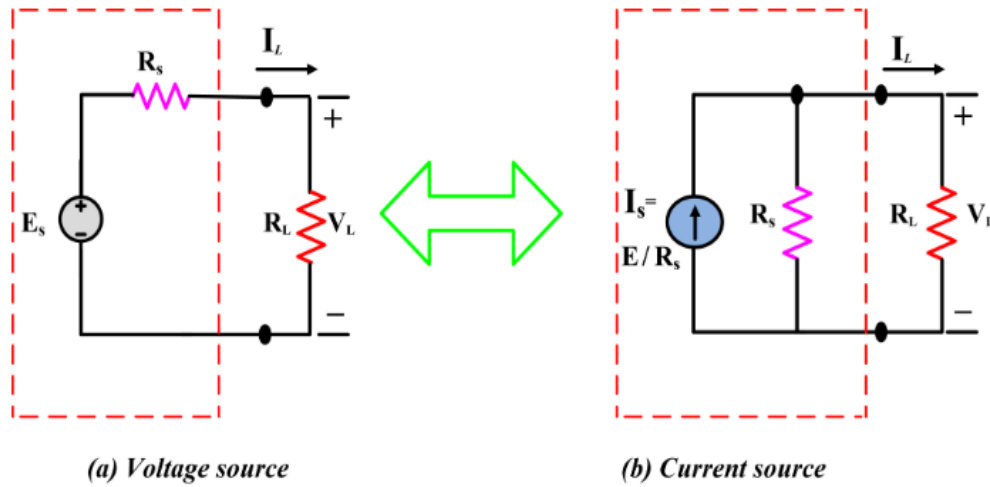

Figure 3.4 Transformation of independent sources.

(a) Voltage source   *(b) Current source*

We need to find the relationship between v_s and i_s that guarantees the two configurations in Fig. 3.4 are equivalent with respect to nodes a, b. Suppose R_L, is connected between nodes a, b in Fig. 3.4(a). Using Ohms law, the current in R_L is.

$$i_L = \frac{v_S}{(R+R_L)} \qquad \text{R and RL in series} \qquad \textbf{(3.5)}$$

If it is to be replaced by a current source then load current must be $\mathbf{V/(R+R_L)}$

Now suppose the same resistor R_L, is connected between nodes a, b in Fig. 3.4 (b). Using current division, the current in R_L, is

$$i_L = i_s \frac{R}{(R+R_L)} \qquad (3.6)$$

If the two circuits in Fig. 3.4 are equivalent, these resistor currents must be the same. Equating the right-hand sides of Eqs.3.5 and 3.6 and simplifying

$$i_s = \frac{v_s}{R} \quad or \quad v_s = i_s R \qquad (3.7)$$

Source transformation also applies to dependent sources, provided we carefully handle the dependent variable. As shown in **Fig. 3.5,** a dependent voltage source in series with a resistor can be transformed to a dependent current source in parallel with the resistor or vice versa.
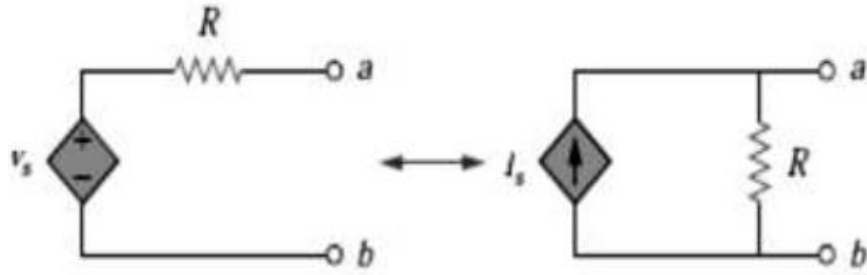
Figure 3.5 Transformation of dependent sources.

However, we should keep the following points in mind when dealing with source transformation.

1. Note from Fig. **3.4 (or Fig. 3.5**) that the arrow of the current source is directed toward the positive terminal of the voltage source.

2. Note from **Eq. (3.7)** that source transformation is **not possible** when **R = 0,** which is the case with an ideal voltage source. However, for a practical, non-ideal voltage source, **R ≠ 0**.

Similarly, an ideal current source with **R =∞** cannot be replaced by a finite voltage source.

Example 3.3: Use source transformation to find **v₀** in the circuit in Fig. 3.6.

Solution:

We first transform the current and voltage sources to obtain the circuit in Fig. 3.7(a). Combining the **4-Ω and 2-Ω resistors in series** and transforming the 12-V voltage source gives us Fig. 3.7(b). We now combine the 3-Ω **and 6-Ω resistors in parallel** to get 2-Ω. We also combine the **2-A and 4-A** current sources to get a 2-A source. Thus, by repeatedly applying source transformations, we obtain the circuit in Fig. 3.7 (c).
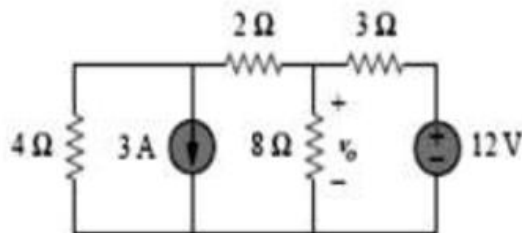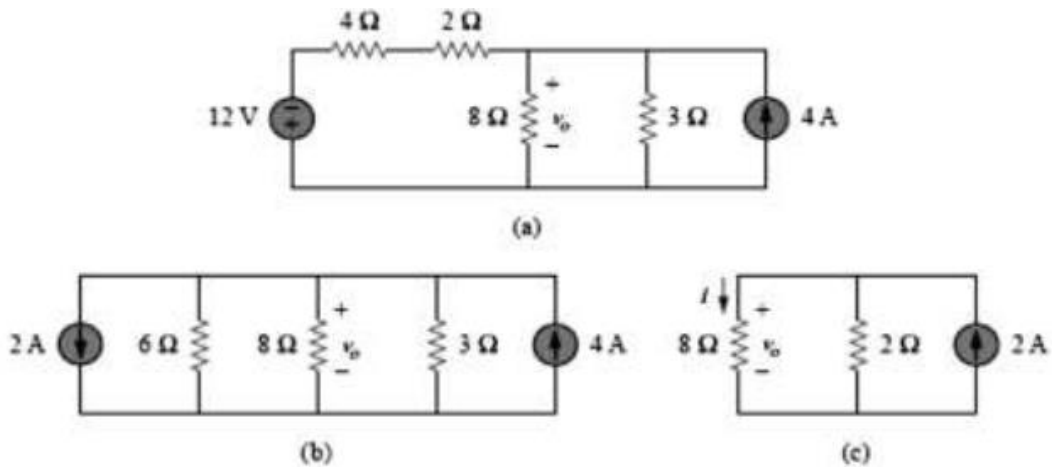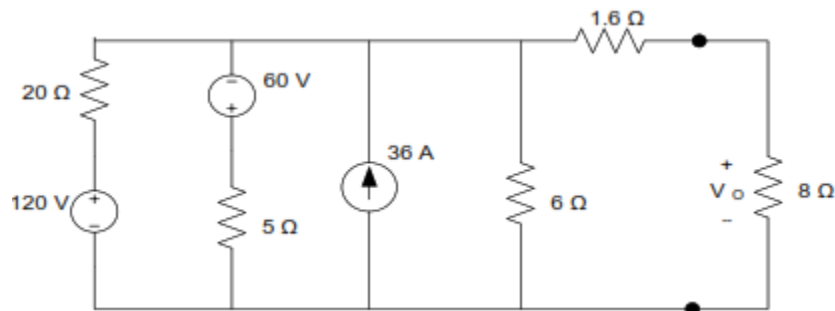

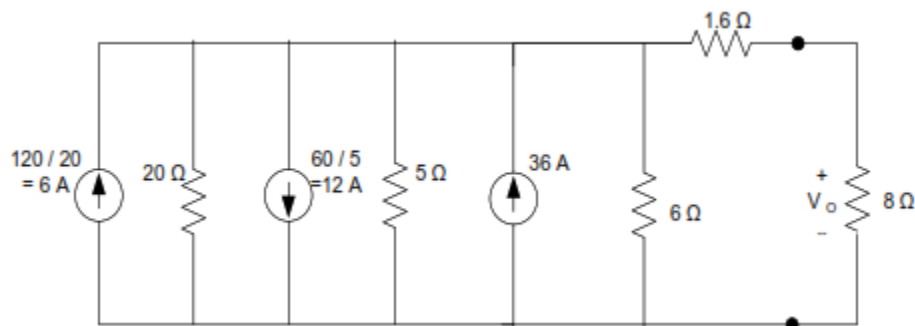Figure 3.6

(a)

(b)   (c)

Figure 3.7

Alternatively, since the 8-Ω and 2-Ω resistors in **Fig. 3.7(c)** are in parallel, they have the same voltage v_o across them. Hence,

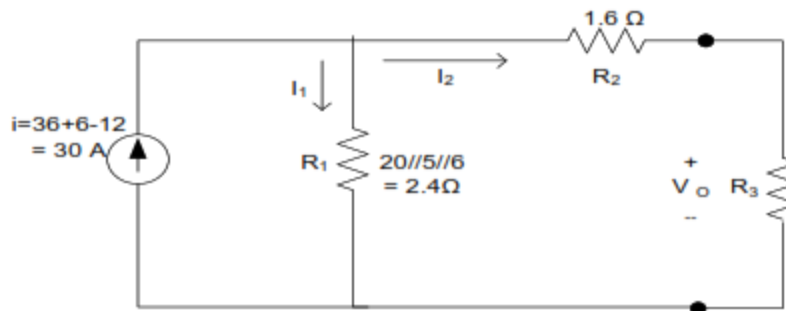$$v_o = (8||2)(2 \text{ A}) = \frac{8\times2}{10}(2) = 3.2 \text{ V}$$

Example 3.3.1: Use source transformation to find **Vo**

Transform the two voltage sources in series with their respective resistances to current sources with parallel resistance
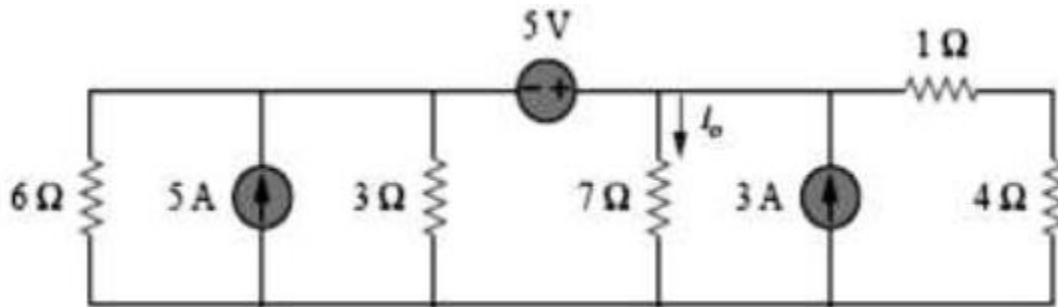

9

$$i_2 = \frac{R_1}{R_1 + R_2 + R_3} \quad i = \frac{2.4}{2.4 + 1.6 + 8}(30) = 6 \text{ A}$$
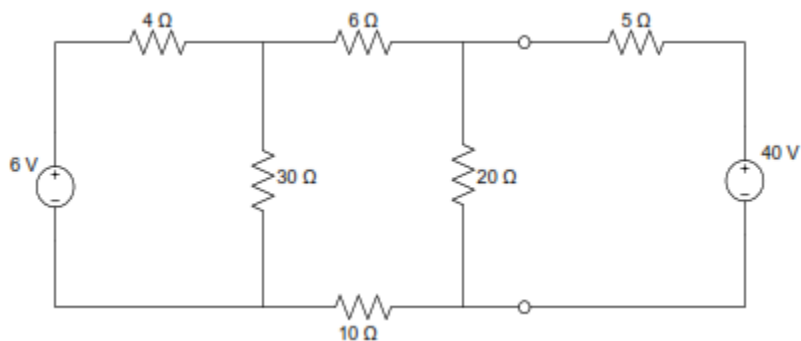
$$V_o = i_2 R_3 = (6 \text{ A})(8\,\Omega)$$

$$V_o = 48 \text{ V}$$

Practice problems:

1-Find i_o in the circuit shown below using source transformation.

2- Using source transformation find the power associated with the 6 V source.

3.5 THEVENIN'S THEOREM

It often occurs in practice that a particular element in a circuit is variable (usually called the load) while other elements are fixed. As a typical example, a household outlet terminal may be connected to different appliances constituting a variable load.

Each time the variable element is changed, the entire circuit has to be analyzed all over again. To avoid this problem, Thevenin's theorem provides a technique by which the fixed part of the circuit is replaced by an equivalent circuit.

According to Thevenin's theorem, the linear circuit in Fig. **3.8(a)** can be replaced by that in **Fig. 3.8(b)** is known as the Thevenin equivalent circuit; it was developed in 1883 by M. Leon Thevenin's (1857– 1926), a French telegraph engineer.

Thevenin's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a voltage source V_{Th} in series with a resistor R_{Th}, Where

- VTh -is the open-circuit voltage at the terminals and
- VTh -Voltage that exists between two port network
- RTh- is the input or equivalent resistance at the terminals when the independent sources are turned off.
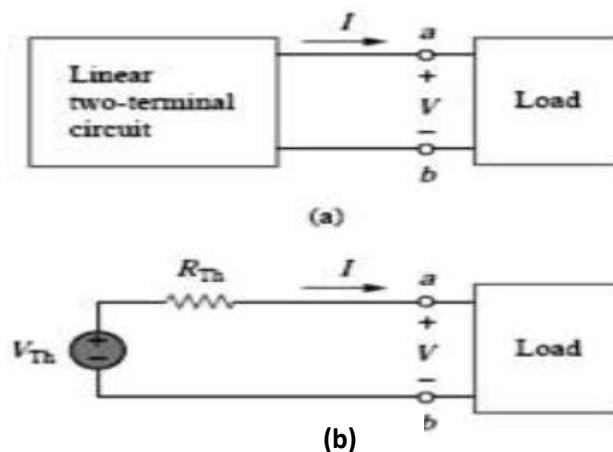

(a)

(b)

Figure 3.8 replacing a linear two-terminal circuit by its Thevenin equivalent: (a) Original circuit, (b) the Thevenin equivalent circuit.

To find the Thevenin equivalent voltage **V$_{Th}$** and resistance **R$_{Th}$,** suppose the two circuits in Fig. 3.8 are equivalent. The open-circuit voltage across the terminals a-b in Fig. 3.8(a) must be equal to the voltage source VTh in Fig. 3.8(b), since the two circuits are equivalent. Thus, V$_{Th}$ is the open-circuit voltage across the terminals as shown in Fig. 3.9(a); that is,
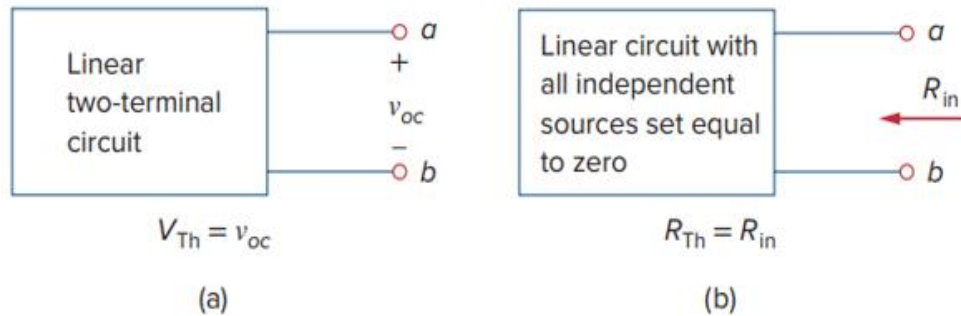
$$V_{Th} = v_{oc} \tag{3.8}$$

Figure 3.9 Finding V$_{Th}$ and R$_{Th}$.

R$_{Th}$ is the input resistance at the terminals when the independent sources are turned off,

as shown in Fig. 3.9(b); that is,

$$\mathbf{R_{Th} = R_{in}} \tag{3.9}$$

To apply this idea in finding the Thevenin resistance RTh, we need to consider two cases.

CASE 1: If the network has no dependent sources, we turn off all independent sources. **R$_{Th}$** is the input resistance of the network looking between terminals **a** and **b,** as shown in **Fig. 3.9(b).**

CASE 2: If the network has dependent sources, we turn off all independent sources. As with superposition, dependent sources are not to be turned off because they are controlled by vo at terminals a and b and determine the resulting current circuit variables. We apply a voltage source i$_o$.

Then R$_{Th}$ = Vo/io, as shown in Fig. 3.10(a).

Alternatively, we may insert a current source io at terminals a-b as shown in Fig. 3.10(b) and find the terminal voltage vo. Again R_{Th} = vo/io. Either of the two approaches will give the same result. In either approach we may assume any value of vo and i_o. For example, we may use v_o = 1 V or i_o = 1 A, or even use unspecified values of **vo or io.**
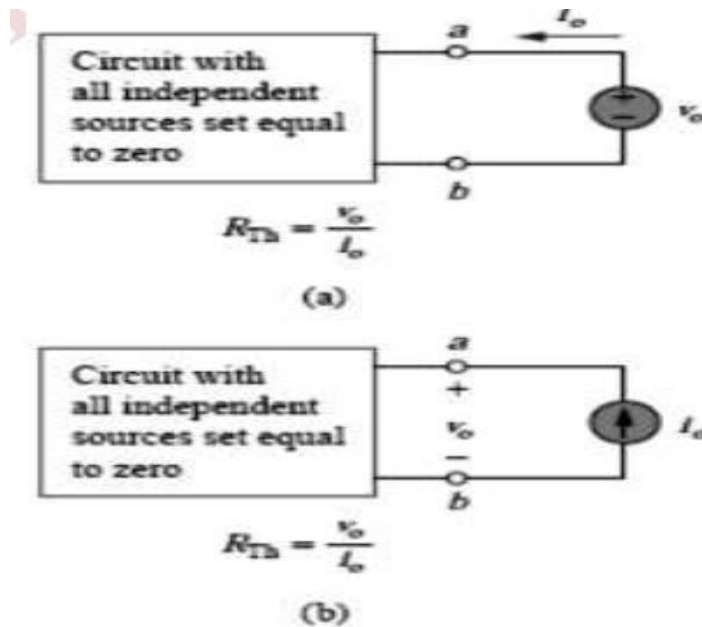

(a)

(b)

Figure 3.10 Finding RTh when circuit has dependent sources.

It often occurs that **RTh** takes a negative value. In this case, the negative resistance (v = −iR) implies that the circuit is supplying power. This is possible in a circuit with dependent sources. The current **IL** through the load and the voltage V_L across the load are easily determined once the Thevenin equivalent of the circuit at the load's Fig. 3.11(b). From Fig. 3.11(b), we obtain

$$I_L = \frac{V_{Th}}{R_{Th} + R_L} \tag{3.10a}$$

$$V_L = R_L I_L = \frac{R_L}{R_{Th} + R_L} V_{Th} \tag{3.10b}$$

Note from Fig. 3.11(b) that the Thevenin equivalent is a simple voltage divider, yielding V_L by mere inspection.
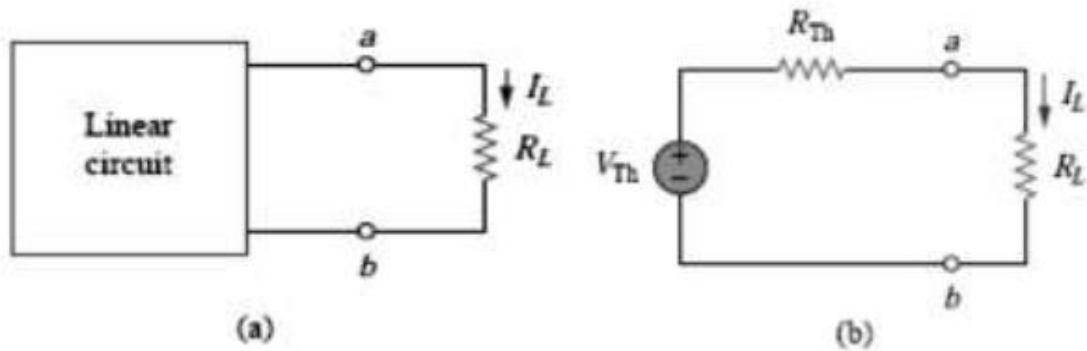
(a)   (b)

Figure 3.11 A circuit with a load :(a) original circuit, (b) Thevenin equivalent.

Example 3.4: Find the Thevenin equivalent circuit of the circuit shown in Fig. 3.12, to the left of the terminals a-b. Then find the current through R_L = 6, 16, and 36 Ω
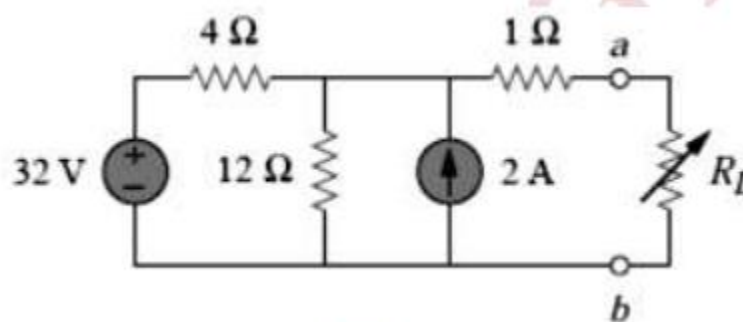

Figure 3.12 for Example 3.4.

Solution:

We find R_{Th} by turning off the 32-V voltage source **(replacing it with a short circuit)** and the 2-A current source **(replacing it with an open circuit)**. The circuit becomes what is shown in Fig. 4.13(a). Thus,

$$R_{Th} = 4||1 + 1 = 4/16 \times 12/ 16 +1 = 4\,\Omega$$

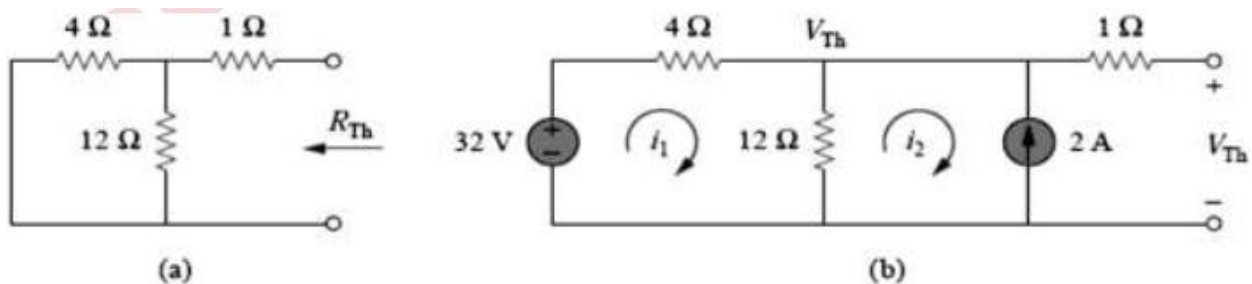(a)   (b)

Figure 3.13 For Example 3.4: (a) finding R_{Th}, (b) finding V_{Th}.

14

To find V_{Th}, consider the circuit in Fig. 3.13(b). **Applying mesh analysis to the two loops, we obtain**

$$-32 + 4i_1 + 12(i_1 - i_2) = 0, \qquad i_2 = -2 \text{ A}$$

Solving for i_1, we get $i_1 = 0.5$ A. Thus,

$$V_{Th} = 12(i_1 - i_2) = 12(0.5 + 2.0) = 30 \text{ V}$$

The Thevenin equivalent circuit is shown in Fig. 3.14. The current through R_L is

$$I_L = \frac{V_{Th}}{R_{Th} + R_L} = \frac{30}{4 + R_L}$$

When $R_L = 6$, $\quad I_L = \frac{30}{10} = 3\ A$

When $R_L = 16$, $\quad I_L = \frac{30}{20} = 1.5\ A$

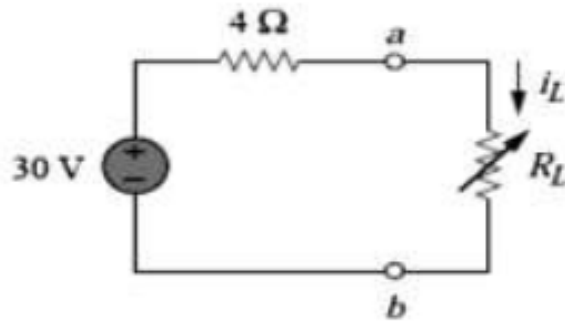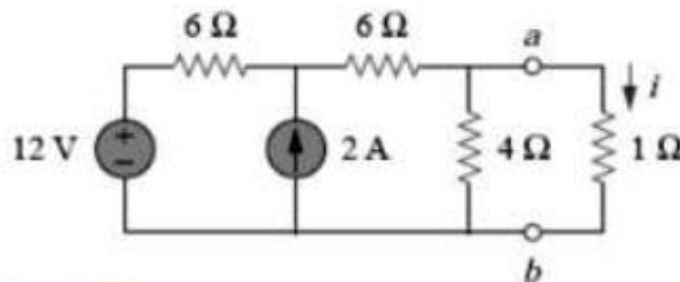When $RL = 36$, $\quad I_L = \frac{30}{40} = 0.75\ A$

Figure 3.14. The Thevenin equivalent circuit

Practice problem:

Using Thevenin's theorem, find the equivalent circuit to the left of the terminals in the circuit in Figure below. Then find i.

3.5 NORTON'S THEOREM

In 1926, about 43 years after Thevenin published his theorem, E. L. Norton, an American engineer at Bell Telephone Laboratories, proposed a similar theorem. Norton's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a current source I_N in parallel with a resistor R_N, where I_N is the short- circuit current through the terminals and R_N is the input or equivalent resistance at the terminals when the independent sources are turned off.

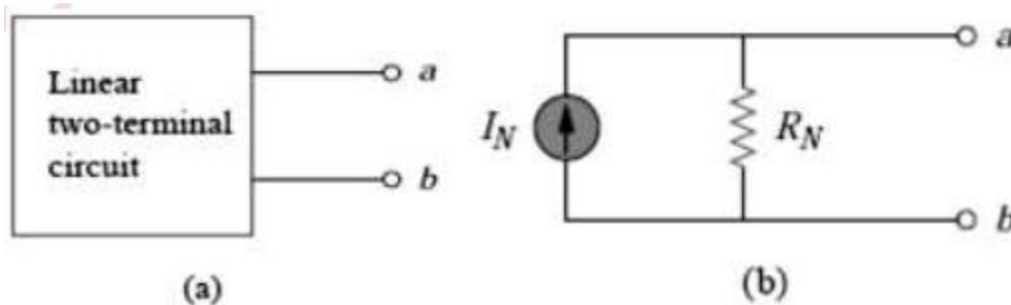Thus, the circuit in **Fig. 4.15(a)** can be replaced by the one in **Fig. 3.15(b).**

(a)   (b)

Figure 3.15 (a) Original circuit, (b) Norton equivalent circuit.

We are mainly concerned with how to get R_N and I_N. We find R_N in the same way we find R_{Th}. In fact, the Thevenin and Norton resistances are equal; that is,

$$R_N = R_{Th} \qquad\qquad\qquad (3.11)$$

To find the Norton current I_N, we determine the **short circuit current flowing from terminal a to b** in both circuits in Fig. 3.15. It is evident that the short-circuit current in Fig. 3.15(b) is I_N. This must be the same short-circuit current from terminal a to b in Fig. 3.15(a), since the two circuits are equivalent. Thus,

$$I_N = i_{sc} \qquad\qquad\qquad (3.12)$$

Dependent and independent sources are treated the same way as in Thevenin's theorem. Observe the close relationship between Norton's and Thevenin's theorems: $R_N = R_{Th}$ as in Eq. (3.11), and

$$I_N = \frac{V_{Th}}{R_{Th}} \tag{3.13}$$

This is essentially source transformation. For this reason, source transformation is often called Thevenin

-Norton Transformation.

We can calculate any two of the three using the method that takes the least effort and use them to get the third using Ohm's law. **Example 4.10** will illustrate this. Also, since

$$\mathbf{V_{Th}} = \mathbf{v_{oc}} \tag{3.14a}$$

$$\mathbf{I_N} = \mathbf{i_{sc}} \tag{3.14b}$$

$$R_{Th} = \frac{v_{oc}}{i_{sc}} = R_N \tag{3.14c}$$

the open-circuit and short-circuit tests are sufficient to find any Thevenin or Norton equivalent.

Example 3.5

Find the Norton equivalent circuit of the circuit in Fig. 3.16.

Solution:

We find $\mathbf{R_N}$ in the same way we find $\mathbf{R_{Th}}$ in the Thevenin equivalent circuit. Set the independent sources equal to zero. This leads to the circuit in Fig. 3.17(a), from which we find $\mathbf{R_N}$. Thus,
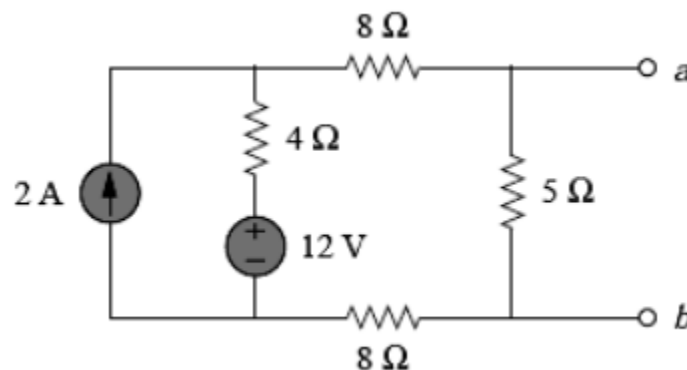

Figure 3.16 For Example 4.5.

To find IN, we short-circuit terminals a and b, as shown in Fig. 3.17(b). We ignore the 5-Ω resistor because it has been short-circuited. Applying mesh analysis, we obtain

$$i_1 = 2 \text{ A}, \qquad\qquad 20i_2 - 4i_1 - 12 = 0$$

From these equations, we obtain

$$i_2 = 1 \text{ A} = i_{sc} = I_N$$

Alternatively, we may determine $\mathbf{I_N}$ from $\mathbf{V_{Th}/R_{Th}}$. We obtain $\mathbf{V_{Th}}$ as the open-circuit voltage across terminals a and b in Fig. 3.17(c). Using mesh analysis, we obtain

$$i_3 = 2 \text{ A}$$

$$25i_4 - 4i_3 - 12 = 0 \quad \Rightarrow i_4 = 0.8 \text{ A}$$

and

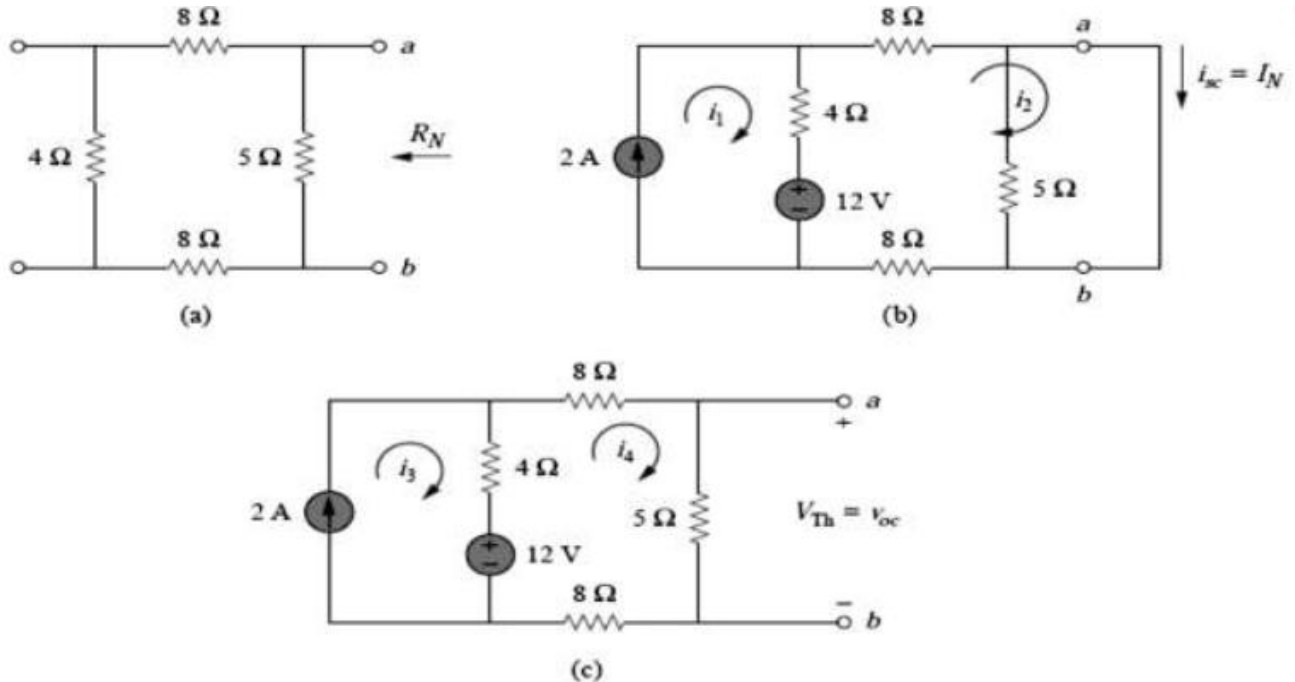$$v_{oc} = V_{Th} = 5i_4 = 4 \text{ V}$$

Figure 3.17 For Example 3.5; finding: (a) R_N, (b) I_N = isc, (c) V_{Th} = v_{oc}.

Hence

$$I_N = \frac{V_{Th}}{R_{Th}} = \frac{4}{4} = 1A$$

as obtained previously. This also serves to confirm **Eq. that R$_{Th}$ = v$_{oc}$/i$_{sc}$ = 4/1 = 4 Ω.** Thus, the **Norton equivalent circuit** is as shown in **Fig. 3.18.**
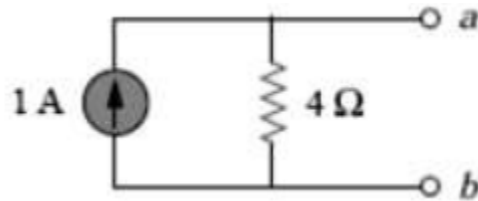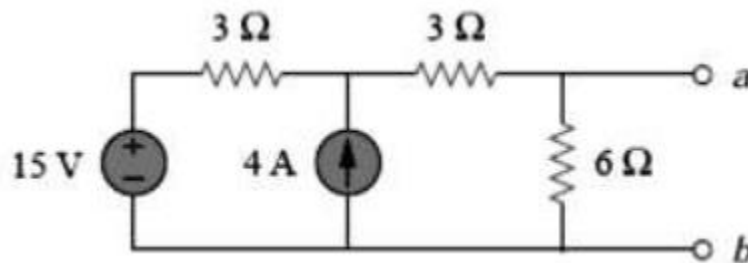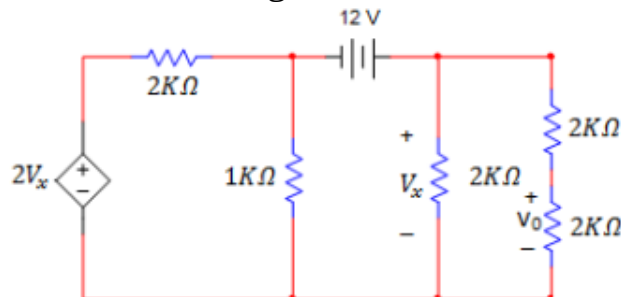
Figure 3.18 Norton equivalent of the circuit in Fig. 3.16.
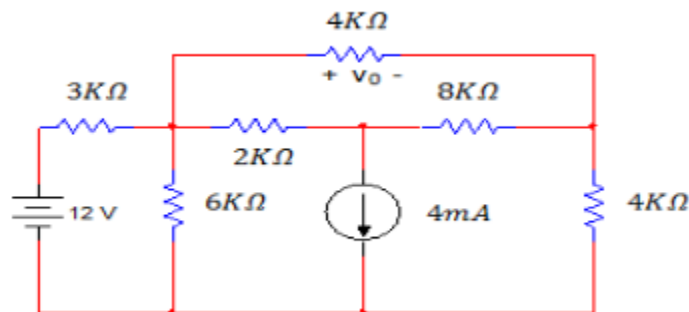
Practice problem:

1- Find the **Norton equivalent circuit** for the circuit in Figure below.

2- Find V$_0$ in the network below using Thevenin's theorem.

3-Find V$_o$ in the network below using Norton's theorem.

3.6 MAXIMUM POWER TRANSFER

In many practical situations, a circuit is designed to provide power to a load. While *for electric utilities, minimizing power losses in the process of transmission and distribution is critical for efficiency and economic reasons, there are other applications in areas such as communications where it is desirable to maximize the power delivered to a load*. We now address the problem of delivering the maximum power to a load when given a system with known internal losses.

The Thevenin equivalent is useful in finding the maximum power a linear circuit can deliver to a load. We assume that we can adjust the load resistance R_L. If the entire circuit is replaced by its Thevenin equivalent except for the load, as shown in Fig. 4.19, the power delivered to the load is
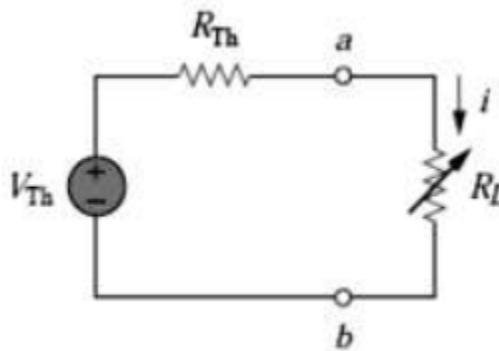

Figure 3.19 The circuit used for maximum power transfer.

$$P = i^2 R_L = (\frac{V_{Th}}{R_{Th} + R_L})^2 R_L \tag{3.15}$$

For a given circuit, **V_Th and R_Th** are fixed. By varying the load resistance **R_L,** the power delivered to the load varies as sketched in **Fig. 4.20**. We notice from **Fig. 4.20** that the power is small for small or large values of **R_L** but maximum for some value of **R_L between 0 and** ∞**.** We now want to show that this maximum power occurs when **R_L** is equal to **R_Th**. This is known as the maximum power theorem.

Maximum power is transferred to the load *when the load resistance equals the Thevenin resistance as seen from the load* **(R$_L$ = R$_{Th}$).**
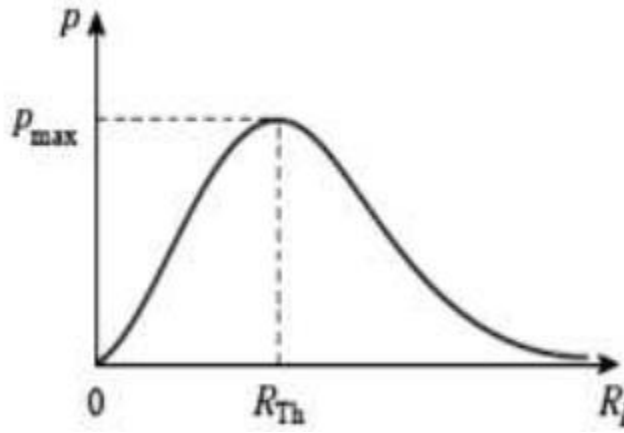

Figure 3.20 Power delivered to the load as a function of R$_L$

To prove the maximum power transfer theorem, we **differentiate p** in **Eq. (3.15)** with respect to **R$_L$** and set the result equal to zero. We obtain

$$\frac{dp}{dR_L} = V_{Th}^2 \left[\frac{(RTh + RL)^2 - 2R_L(R_{Th} + R_L)}{(R_{Th} + R_L)^4}\right] = V_{Th}^2 \left[\frac{(R_{Th} + R_L - 2R_L)}{(R_{Th} + R_L)^3}\right] = 0$$

This implies that

$$0 = (R_{Th} + R_L - 2_{RL}) = (R_{Th} - R_L) \tag{3.16}$$

Which yields

$$R_L = R_{Th} \tag{3.17}$$

Showing that the **maximum power transfer takes place when the load resistance R$_L$ equals the Thevenin resistance R$_{Th}$.** We can readily confirm that **Eq. (3.17)** gives the maximum power by showing that **d^{2}p/dR$_L^2$ <0.** The maximum power transferred is obtained by substituting **Eq. (3.17) into Eq. (3.15),** for

$$P_{max} = \frac{V_{Th}^2}{4R_{Th}} \tag{4.18}$$

Equation (4.18) applies only when **R$_L$ = R$_{Th}$.** When **R$_L$ ≠ R$_{Th}$**, we compute the power delivered to the load using **Eq. (3.15).**

Example 3.6:

Find the value of R_L for maximum power transfer in the circuit of Fig. 3.21. Find the maximum power.
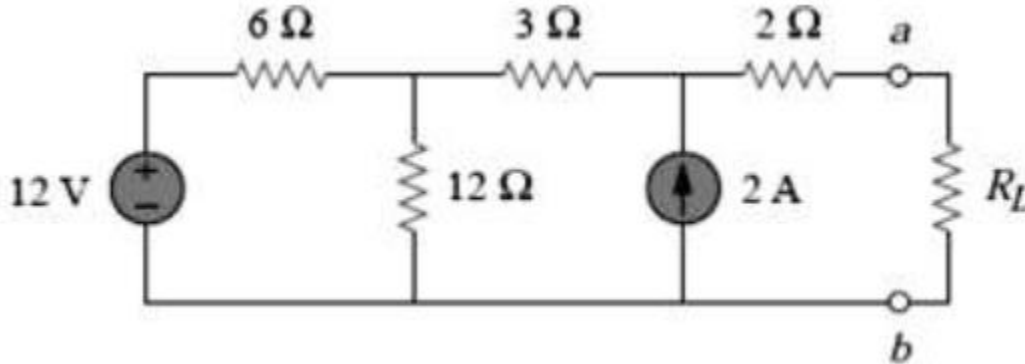

Figure 3.21 For Example 3.6.

Solution:

We need to find the Thevenin resistance R_{Th} and the Thevenin voltage V_{Th} across the terminals a-b. To get R_{Th}, we use the circuit in **Fig. 3.22(a)** and obtain

$$R_{Th}= 2 + 3 + 6\|12 = 5 + 6\text{x}12/18 = 9 \ \Omega$$

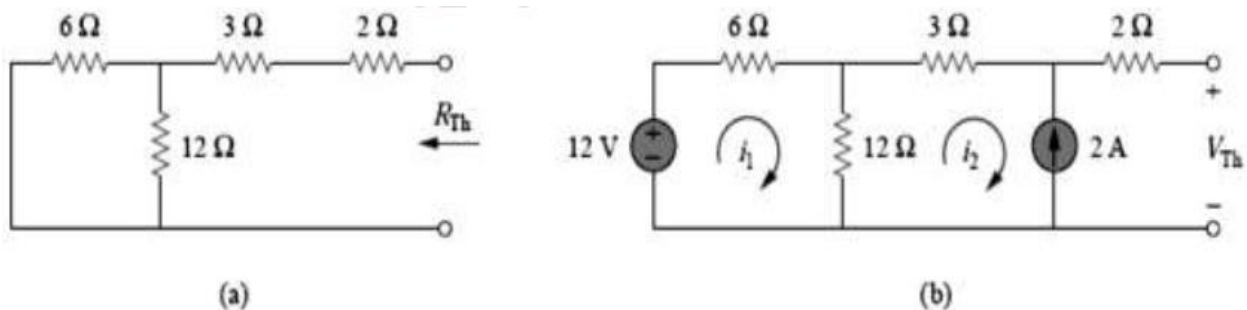(a)   (b)

Figure 3.22 For Example 3.6: (a) finding R_{Th}, (b) finding V_{Th}.

To get V_{Th}, we consider the circuit in Fig. 3.22(b). Applying mesh analysis,

$$-12 + 18i_1 - 12i_2 = 0, \qquad i_2 = -2 \text{ A}$$

Solving for i_1, we get $i_1 = -2/3$. Applying KVL around the outer loop to get V_{Th} across terminals a-b, we obtain

$$-12 + 6i_1 + 3i_2 + 2(0) + V_{Th} = 0 \Rightarrow V_{Th} = 22 \text{ V}$$
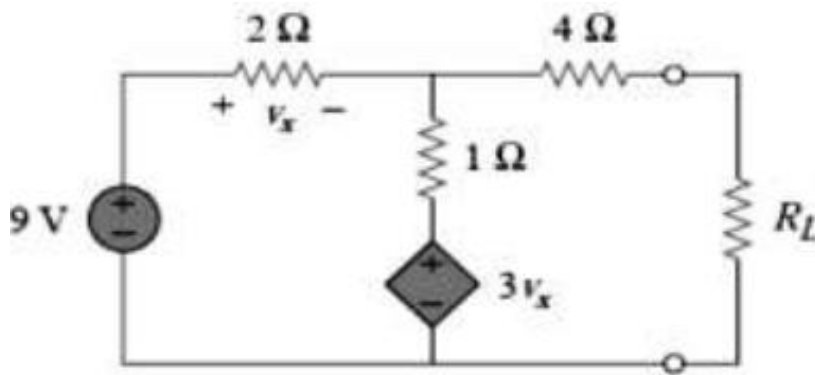
For maximum power transfer,

$$R_L = R_{Th} = 9\ \Omega$$

and the maximum power is

$$P_{max} = \frac{V^2_{Th}}{4R_L} = \frac{22^2}{4 \times 9} = 13.44W$$

Practice problem:

1-Determine the value of R_L that will draw the maximum power from the rest of the circuit in Figure below. Calculate the maximum power.

2-Find the value of R_2 for maximum power transfer in the network below and the maximum power that can be transferred to this load.